THE

MATHEMATICAL GAZETTE.

EDITED BY

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WITH THE CO-OPERATION OF

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COMMITTEE ON THE TEACHING OF ELEMENTARY MATHEMATICS.

A meeting of the above Committee of the Mathematical Association was held at King's College, London, on Saturday, Oct. 3rd, the President, Prof. A. R. Forsyth, F.R.S., being in the chair.

Messrs. C. S. Jackson (Royal Military College, Woolwich) and L. S. Milward (Malvern College) were elected members of the Committee.

It was decided to send a letter to the Head Masters of Public Schools asking how far they would recognise and insert questions in Practical and Theoretical Geometry in their Entrance Examinations and Scholarship Entrance Examinations, in accordance with the syllabus recommended by the Cambridge Syndicate, which was adopted by Grace of the Senate on June 11th, 1903. The object of seeking this information is to afford guidance to Preparatory Schools in the teaching of Geometry.

A Sub-Committee was formed to consider and report on the teaching of Elementary Mechanics. Five members were elected, with power to add to their numbers, viz.: Prof. G. M. Minchin, and Messrs. F. W. Hill, C. S. Jackson, A. W. Siddons, and C. O. Tuckey. Communications in reference to the work of this Sub-Committee should be addressed for the present to C. O. Tuckey, Esq., M.A., Charterhouse School, Godalming.

M. A. COMMITTEE.

THE following letter has been addressed to all the schools mentioned in the *Public Schools Yearbook*.

TRINITY COLLEGE, CAMBRIDGE,
October 19th, 1903.

Dear Sir,—The Committee of the Mathematical Association has been approached by the Association of Headmasters of Preparatory Schools as to the advisability of at once beginning to teach Geometry in Preparatory Schools on the lines laid down in the new regulations for the Cambridge Previous Examination.

It may be pointed out that these regulations come into full force after 1905 (during 1904 and 1905 examinations will be held under both new and old regulations); and the new regulations at Oxford come into force in the autumn of 1904. That being so, all boys at present at Preparatory Schools

who proceed to the Universities must of necessity be examined there under the new regulations.

In these circumstances, many Preparatory School Masters feel that they ought to take steps to introduce the new methods at once; but they are not confident that, if they do so, they may not place some of their pupils at a disadvantage for a short time in the immediate future.

In order that these School Masters may be in a position to act upon fuller knowledge, I shall be much obliged if you will be kind enough to answer the questions below, and to communicate your answers to the Honorary Secretary of the Committee, A. W. Siddons, Esq., Harrow-on-the-Hill.

If you have printed regulations for your entrance, and entrance scholarship examinations, I should be glad if at the same time you would kindly send Mr. Siddons a copy, in order that the Committee may be able to refer to them from time to time.—I am, Yours faithfully,

A. R. Forsyth, *President, Math. Assoc.*

P.S.—A copy of this letter is being sent to your Senior Mathematical Master.

Name of School.....

1. Will questions on experimental and practical geometry be set in your entrance and entrance scholarship examination?

2. Will the freedom of proof and method set out in the new University regulations be allowed in these examinations?

3. Will boys, after admission to your School, be restricted to Euclid in method and matter?

Replies have been received from the following schools:

Abingdon, Aldenham, Bath, Bedford, Berkhamsted, K.E.S. Birmingham, Blundell's (Tiverton), Birkenhead, Bradfield, Bradford, Bromsgrove, Cambridge (Leys and Perse Schools), Canterbury (King's School and St. Edmund's), Carlisle, Charterhouse, Cheltenham (College and Dean Close School), Chester, Chigwell, City of London, Clifton, Merchant Taylors' Crosby, Derby, Dover, Dulwich, Durham, Eastbourne, Epsom, Eton, Exeter, Felsted, Giggleswick, Glenalmond, Guernsey, Haileybury, Harrow, Hereford, Highgate, Hull, Ipswich, King William's College, Isle of Man, Jersey, King's College School, Lancing, Leeds, Liverpool College, Malvern, Manchester, Marlborough, Stroud, Merchant Taylors, Merchiston Castle, Mill Hill, Monmouth, Newcastle, Norwich, Oakham, Oundle, Oxford (St. Edward's), Plymouth, Pocklington, Portsmouth, South Eastern College, Ramsgate, Reading, Repton, Rossall, Rugby, St. Bees, St. Olave's, Sedbergh, Sherborne, Shrewsbury, Stonyhurst, Sutton Valence, Tonbridge, Trent College, University College School, Uppingham, Wakefield, Warwick, Wellington, Westminster, Winchester, Worcester.

Seventeen schools have not yet replied.

To question 1, the reply was in each case in the affirmative except as follows: Clifton, Derby and Durham answered "probably"; St. Olave's "possibly"; Oakham "will set questions on practical but not on experimental, Merchiston and Liverpool do not include Geometry in these examinations.

To question 2, the reply was "yes" in every case except that of Cheltenham College. They say "Freedom of proof and method allowed if it does not violate the logical sequence of Euclid."

To question 3, the replies were all in the negative, though at one school Euclid is still retained as a textbook, and a selection of propositions is made.

Berkhamsted, instead of answering the separate questions, said that Euclid was entirely abandoned.

The replies show how very real a reform is being made; every school seems to encourage practical work, and, except in one case, to grant the

freedom which the Universities have considered it wise to give. The report of the Committee has played no small part in obtaining these reforms; and though, at the time of publishing the report, it seemed politic not to advocate quite so much freedom, the Committee warmly welcomes the additional freedom granted by the Universities.

A. W. SIDDONS, *Hon. Sec. Math. Assoc. Committee.*

TO REACH THE CALCULUS AS EARLY AS POSSIBLE.

In these days the student of applied science finds that he can make little progress till he has acquired a knowledge of the Calculus. If he cannot acquire this knowledge sufficiently soon in the mathematical class-room, his science teacher will have to do the work which ought to fall to the duty of the mathematician, and will at the same time be thereby debarred from devoting the time to other teaching which he would like to give.

There are many things in mathematics—*e.g.*, that much over-rated landmark in the construction of syllabuses, the Binomial Theorem*—which are of much less use to such a student than a mere knowledge of the notation of the Calculus. The following remarks are put forward as suggestions of how the Calculus might be commenced at an earlier stage in the study of algebra and trigonometry than has hitherto been the practice. They may at least serve as a basis for further discussion.

I. *Algebraic Forms.*

(1) As soon as a beginner has learnt a very little about multiplication and division in algebra, he may be taught the *proper* way (not the way he usually uses) to divide $(x^2-3x+2)-(a^2-3a+2)$ by $x-a$. By putting x for a in the quotient the differential coefficient of x^2-3x+2 is obtained, and in this way the student may learn to find the differential coefficients of simple algebraic forms.

(2) The differential coefficient of a positive integral power can be found as soon as the beginner has learnt to divide x^n-a^n by $x-a$. When this has been done, easy instructive exercises may be given in writing down the differential coefficients of such expressions as $x^5-16x^4+5x^3+2x^2+x-1$, and the expressions of which these are the differential coefficients, even without previously having wasted time in multiplying or dividing such expressions, or in finding their greatest common measure or least common multiple.

Greatest common measure in algebra has always seemed to me a useless piece of drudgery. The sums take a long time to work out and often come wrong, and I have so far failed to discover any case in which the method is actually made use of, except in connection with Sturm's Theorem on the roots of a quantic equation.

(3) The differential coefficient of $1/x^n$ is equally easy to obtain, and may be discussed in connection with the meaning of negative integral indices. The interpretation of these may easily be arrived at by continuing backwards the processes which give rise to the series x, x^2, x^3, x^4, etc.

(4) The differential calculus may be used with advantage to determine the coefficients in the expansion of $(a+x)^n$, and thus to prove the Binomial Theorem for a positive integral index. The method will be identical with that employed in the ordinary determination of the coefficients in Maclaurin's Theorem. This proof is at any rate as short and simple as, and

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* The Binomial Theorem is merely the formula for shifting the origin in the graph of $y=x^n$.

if anything more direct and straightforward than, the usual proofs involving permutations and combinations or mathematical induction.

(5) As soon as fractional indices are introduced in algebra, expressions involving them may be differentiated, the differential coefficient of $x^{p/q}$ being deduced from the limiting value of $(z^p-a^p)\div(z^q-a^q)$.

(6) The Binomial Theorem for a fractional or negative index may well be deduced from Taylor's Theorem. This theorem not only enables us to find the expansion, but to write down the remainder after any number of terms. We are thus relieved from giving the faulty demonstrations of the older text-books, in which a theorem was proved to be true, which it was afterwards shown could not possibly be true, or from giving a lengthy discussion of theorems relating to the product of two absolutely convergent series, which are of little interest to the student of experimental science.

Having read a number of examination answers, in which the candidates commenced their proofs of the Binomial Theorem with the brazen-faced assertion that $f(m)\times f(n)=f(m+n)$, and in some cases stated that $(1+x)^{p/q}=(1+x)\times(1+x)\times\ldots$ to p/q factors, I do not think matters could be worse than they are at present. I believe that the employment of Taylor's Theorem in the proof of the Binomial Theory would be as well understood as the present cumbersome proofs, and would involve less waste of time in the case of the science students.

II. *Exponential and Logarithmic Expressions.*

(1) If the Binomial Theorem is proved by means of Taylor's Theorem, the consideration of exponential and logarithmic *series* is naturally deferred till this stage.

The calculus up to and including Taylor's Theorem may be adequately illustrated by algebraic forms, excluding exponential and logarithmic forms, which may come later. But a treatment of these forms need not of necessity require the previous knowledge of the Binomial Theorem, as we shall now show.

(2) If we define e to be the limit, when z is infinite, of $(1+1/z)^z$, *and we make the assumption that* 'e' *is a finite quantity greater than unity*, we immediately deduce that if l is the limit of $(a^h-1)/h$ when h is zero, then $e^l=a$, or $l=\log_e a$. This enables us to differentiate all exponential and logarithmic forms and prove their properties.

(3) As it does not appear easy to prove the above assumption as to the nature of e without using the Binomial Theorem, the following alternative treatment may be suggested: By the definition of a differential coefficient it follows that

$$\frac{1}{a^x}\frac{d(a^x)}{dx}=\operatorname*{Lt}_{h=0}\frac{a^h-1}{h}=l \text{ say.}$$

By means of a graph of a^x or otherwise, it may now be shown (a) that l is finite when a is finite, (b) that l is positive or negative according as a is greater or less than unity, and (c) that l increases as a increases.

But further, putting $z=1/hl$ in the above definition, we get

$$a^{1/l}=\operatorname*{Lt}_{z=\infty}\left(1+\frac{1}{z}\right)^z.$$

Hence from (a), (b), (c), it follows that e is a finite positive quantity greater than unity.

(4) Or again, defining $\log_e a$ as being the limit

$$\operatorname*{Lt}_{h=0}(a^h-1)/h,$$

it may be easily shown that $\log_e a+\log_e b=\log_e(ab)$, and hence that quantities

like $\log_a a$ satisfy the ordinary known properties of logarithms. In order to avoid any semblance of preconceived ideas being assumed as to the logarithmic nature of these limits the Continental notation "lognat a" might be preferable to $\log_a a$.

III. *Trigonometrical Expressions.*

(1) There is no point in introducing radian measure as long as only the properties of finite angles are considered. But as soon as the sum or difference of two sines or cosines have been factorised, circular measure may be introduced and immediately applied to the differentiation and integration of trigonometric functions.

(2) The expansions of the sine and cosine may be then deduced from Taylor's or Maclaurin's Theorem, which will have been studied in algebra by the time the student reaches this stage.

IV. *Geometrical Applications.*

(1) The Calculus might be introduced into geometry at a much earlier stage than is now done, especially in connection with tangents to curves.

(2) This is particularly the case in connection with conics. In finding the equations to the tangents to conics, the use of the Calculus would save students from learning up separate proofs which are really of no after use.

(3) As a further instance, the bifocal property of the tangent to a conic might be deduced by means of the property that if $r+r'=2a$, then $dr/ds+dr'/ds=0$, and conversely this property would illustrate the meaning of dr/ds and dr'/ds. Such illustrations would revive interest in the study of conics. This subject is at present not a popular one with students.

(4) The uninteresting chapter in our text-books on "Polar equation of a conic about a focus" would become much more useful and interesting if the results were applied to illustrate "tangents and areas of polar curves" treated by the Calculus.

Much more might be said in connection with the early introduction of the Calculus into geometry. My object in writing these notes has been, however, to consider more especially its early introduction into algebra, as the teaching of algebra has not received so much attention of late as that of geometry, and it is, to my mind, *even more* urgently in need of reform.

Conclusion.

The necessity of bringing the study of the Calculus nearer to the front in mathematical teaching appears to me to tend in the direction of *abolishing the study of the Calculus as a separate subject.* In other words, a text-book on algebra will make free use of the Calculus as applied to algebra; a text-book on trigonometry will make free use of the Calculus as applied to trigonometry, and so on. This will only bring such books into uniformity with those on the applied sciences, where a text-book on physics makes use of the Calculus as applied to physics, and one on chemistry makes use of the Calculus as applied to chemistry.

The Calculus seems to appeal to and to interest the average student much more than a great deal of the subject matter commonly included in a mathematical course.

As an example, it is not uncommon to find that when a course of lectures has been given on conics for two terms, followed by a very short course on the Calculus, students attending the classes will, when examined on the work, answer the Calculus questions well and completely fail in their conics.

G. H. Bryan.

THE USE OF TANGENTIAL COORDINATES.

THERE are two reasons why the use of tangential coordinates should be introduced at an earlier stage than is customary into the teaching of analytical geometry and placed on a somewhat different footing. The educational advantages of encouraging the student to follow out a course of reasoning with the help of only a few symbols instead of grinding out hard examples by long meaningless algebra should not require much pressing, although they do not seem to be sufficiently recognised in this country; and on the other hand many theories receive their most appropriate expression in tangentials chiefly owing to the fact that the absolute has a less specialised form in tangential than in point coordinates, being (in two dimensions) the product of two distinct factors in the former and a perfect square in the latter.

The student must, of course, begin with point coordinates and will at first regard an equation in line coordinates as the condition for an envelope: but gradually the new idea of regarding a line as a fundamental geometrical entity must be introduced, and the ultimate goal is the ability to *read* a tangential equation without any preliminary translation into point coordinates.

To take an example from the elementary part of the subject, it is an easy exercise to express the general tangential equation of a circle in the form

$$k(l^2+m^2)+(Gl+Fm+C)^2=0,$$

from which the whole of the interesting projective theory can clearly be deduced. Again the condition for a parabola, $C=0$, is at the same time simpler and of more interesting interpretation than the more familiar condition $ab=h^2$.

Turning now to more advanced subjects, the connection between tangential equations, complex variables, and foci is instructive. Each of the equations

$$la+mb+n=0$$

$$a+ib-(x+iy)=0$$

is sufficient to define the point (a, b). The latter is the equation of the line joining (a, b) to one of the circular points at infinity. If this line is a tangent to a real curve, the point (a, b) is a focus. We therefore can obtain the foci of any curve by substituting $l=1$, $m=i$, $-n=z=x+iy$ in the tangential equation and solving the resulting equation for the complex variable z. Conversely, for example, if

$$f(z)\equiv(z-z_1)(z-z_2)(z-z_3), \quad z_s=a_s+ib_s,$$

then the roots of

$$f'(z)=0$$

are the foci of the conic

$$u_2u_3+u_3u_1+u_1u_2=0, \quad u_s=la_s+mb_s+n,$$

which touches the sides of the triangle $z_1z_2z_3$ at their middle points. This theorem, of which other proofs have been given, is evidently capable of generalisation to polynomials of any degree.

An interesting theorem and one immediately proved from the tangential equation, though offering considerable difficulties when treated otherwise, is concerned with the 'centre' of an algebraic curve.* This point is best defined as the polar point of the line at infinity, and the theorem states that it is the centroid of the points of contact of parallel tangents. Let the tangential equation of a curve of class μ be

$$\phi(l, m, n)\equiv n^\mu+(al+bm)n^{\mu-1}+\ldots=0.$$

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* See *Quarterly Journal*, vol. xxiv, 1890, Taylor, p. 55, and Baker, p. 338.

The polar curve of class ν of the line (l', m', n') is

$$\left(l'\frac{\partial}{\partial l}+m'\frac{\partial}{\partial m}+n'\frac{\partial}{\partial n}\right)^{\mu-\nu}\phi(l, m, n)=0.$$

and hence the polar point of the line at infinity has for its equation

$$al+bm+\mu n=0$$

representing the point $(a/\mu, b/\mu)$. For given values of l and m the equation $\phi=0$ gives μ values of n whose sum is $-(al+bm)$; hence the line of mean position among μ parallel tangents is

$$lx+my-\mu^{-1}(al+bm)=0,$$

passing through the centre $(a/\mu, b/\mu)$ which is independent of l/m. From the consideration that two near tangents intersect in a point which becomes ultimately the point of contact the theorem follows as stated.

Great clearness is introduced into the theory of averages in connection with areas and volumes by the exclusive use of tangential coordinates. In two dimensions it is best to lead up to the theory of moments by beginning with the *null conic*, that is, the envelope of lines about which the area or lamina has no second moment. Leaving out the density for simplicity—the modifications due to variable density are easily introduced—the equation of the null conic is

$$\int\int(lx+my+n)^2\,dx\,dy=0.$$

It is often desirable to omit the signs of integration, and to substitute for them the notation of averages. Thus, by a proper choice of axes the above may be written

$$l^2\overline{x^2}+m^2\overline{y^2}+n^2=0.$$

Then the *ellipse of gyration* is the confocal

$$l^2\overline{y^2}+m^2\overline{x^2}-n^2=0,$$

and the *ellipse of inertia* is the conic 'conjugate' to the null conic

$$l^2\overline{x^2}+m^2\overline{y^2}-n^2=0.$$

One last example will suffice to illustrate the advantage of attacking certain problems by tangentials from the beginning. The *surface of floatation* is the envelope of a plane cutting off a constant volume from a vessel. We want its approximate equation, correct as far as the curvatures. In this, as in other approximate work, it is convenient to take as the standard equation of a plane

$$z+n=lx+my,$$

for then, when x, y, l, m are small quantities of the first order, z and n can be regarded as of the second order.

Taking a vessel of any shape, the condition of cutting off a constant volume, which is itself the tangential equation of the envelope, is, as far as terms of the second order,

$$\int\int z\,dx\,dy+\int\tfrac{1}{2}z\,\delta\nu\,ds=0,$$

where z has the above value: the integrals are taken over and round the section of floatation, and at the edge, $z/\delta\nu$ is the tangent of the angle between the side of the vessel and the section. Let δz be any small constant and let $\delta\nu_1/\delta z=\delta\nu/z$; then

$$\int z\,\delta\nu\,ds=(\delta z)^{-1}\int z^2\delta\nu_1\,ds=\frac{\delta}{\delta z}\int\int z^2\,dx\,dy$$

where the symbol δ refers to a change to a parallel section at distance δz. Hence the equation of the surface is

$$\iint (lx+my-n)dxdy+\frac{1}{2}\frac{\delta}{\delta z}\iint (lx+my-n)^2dxdy=0,$$

and when the centroid of the section is the origin, this takes the elegant form

$$2n+\frac{\delta}{\delta V}\iint (lx+my)^2dxdy=0,$$

V being the volume immersed. This form gives the curvature immediately, and renders clear the connection with the momental ellipse of the section and with the surface of buoyancy.*

R. W. H. T. Hudson, M.A., D.Sc.

St. John's College, Cambridge.

NOTE ON THE TREATMENT OF CONIC SECTIONS AND CONICOIDS BY PURE GEOMETRY.

The object of the following note is to derive Pascal's theorem and the Anharmonic and Harmonic properties of Conic Sections directly from the focus and directrix definition by the early introduction of an extended form of Carnot's theorem, viz. "the product of the ratios in which the sides of a closed polygon, taken in order, are cut by a conic $=+1$." When the above have been demonstrated the simpler properties are known to be easily deducible. Subsequently the treatment of Conicoids by Pure Geometry is introduced by means of the same theorem.

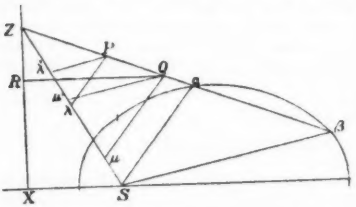
The sequence of propositions, commencing after the definition of a Conic, is as follows (proofs where necessary are very briefly given):

(1) Construct points on the curve, and observe that the curve can nowhere cease abruptly, so that any branch it may have must either be closed or extend to infinity at each end.

(2) If a conic meets a line at α and β, and P, Q are two other points on that line, the product of the ratios in which PQ is cut at α and β

$$=\frac{SP^2-e^2PM^2}{SQ^2-e^2QN^2},$$

where PM, QN are perpendicular to the directrix.



Let PQ meet the directrix at Z.

Draw $P\lambda$, $Q\mu$ parallel to $S\alpha$ and $P\lambda'$, $Q\mu'$ parallel to $S\beta$ to meet SZ.

Draw αR perpendicular to directrix.

Then $$\frac{P\lambda}{S\alpha}=\frac{ZP}{Za}=\frac{PM}{\alpha R}; \quad \therefore\ P\lambda=ePM.$$

So $$P\lambda'=ePM, \text{ and } Q\mu=eQN=Q\mu'.$$

Also $$\frac{P\alpha}{\alpha Q}\cdot\frac{P\beta}{\beta Q}=\frac{S\lambda}{S\mu}\frac{S\lambda'}{S\mu'}=\frac{SP^2-e^2PM^2}{SQ^2-e^2QN^2},$$

since λ and λ' lie on a circle whose centre is P and radius ePM.

* Hudson, *Messenger of Mathematics*, vol. xxxiii, p. 50.

COROLLARY. A conic meeting a line in a point α meets it in another point β which may however coincide with α: but it cannot meet it in any other point γ: for in that case $\frac{P\alpha}{\alpha Q}\cdot\frac{P\beta}{\beta Q}$ would $=\frac{P\alpha}{\alpha Q}\cdot\frac{P\gamma}{\gamma Q}$, or $\frac{P\beta}{\beta Q}=\frac{P\gamma}{\gamma Q}$, numerically, for all positions of P and Q on the line.

(3) If every side of a closed polygon $A_1A_2\ldots A_n$ meets a conic in two points, either separate or coincident, then the product of the ratios in which the sides of the polygon, taken in order, are cut by the curve $=1$.

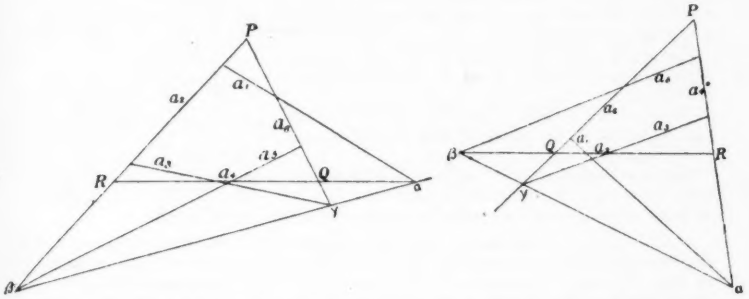
If A_1M_1, $A_2M_2\ldots A_nM_n$ are perpendicular to the directrix, the numerical value of the product

$$=\frac{SA_1^2-e^2A_1M_1^2}{SA_2^2-e^2A_2M_2^2}\cdot\frac{SA_2^2-e^2A_2M_2^2}{SA_3^2-e^2A_3M_3^2}\cdots\frac{SA_n^2-e^2A_nM_n^2}{SA_1^2-e^2A_1M_1^2}=1.$$

As regards sign: in tracing the polygon the pencil starts at A_1 and returns to A_1. It must therefore cross each branch of the conic an even number of times, since by (1) any branch is closed or extends to infinity at each end. Hence the sides of the polygon are cut internally, and therefore externally, in an even number of points, so that the above product is positive.

(4) Pascal's theorem.

Let the sides of an inscribed hexagon be called $a_1, a_2, a_3, a_4, a_5, a_6$, of which a_2, a_4, a_6 form a $\triangle PQR$. Let α, β, γ be the meets of $a_1, a_4 : a_2, a_5 : a_3, a_6$.



Then the sides of the $\triangle PQR$ are cut in 9 points by the 3 lines a_1, a_3, a_5, and the product of the 9 ratios in which QR, RP, PQ are cut $=-1$, by Menelaus' theorem.

And the product of the ratios in which they are cut at the vertices of the hexagon $=1$, by (3);

$$\therefore\ \frac{Q\alpha}{\alpha R}\cdot\frac{R\beta}{\beta P}\cdot\frac{P\gamma}{\gamma Q}=-1,$$

so that α, β, γ are collinear.

(5) If A, B are fixed points on a conic and P a moveable point on the same,

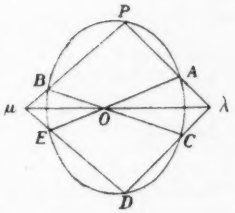
$$A(P)=B(P).$$

Let C, D, E be any fixed pts. on the conic.

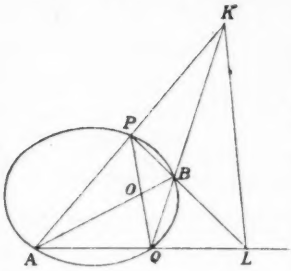
Let CD meet AP at λ, DE meet PB at μ, and AE meet BC at O.

Then $\lambda O\mu$ is the Pascal line of $PAEDCB$;

$$\therefore\ A(P)=(\lambda)=(\mu)\ \text{(since } O \text{ is fixed)}$$
$$=B(P).$$



(6) The locus of the meets of tangents at the extremities of a chord PQ which passes through a fixed point O is a straight line which is also the locus of the harmonic conjugate of O with respect to P and Q.



Let AOB be any fixed chord through O.

Let AP meet BQ at K and PB meet AQ at L.

Then KL is the Pascal line of a hexagon formed by the sides of $APBQ$ and the tangents at P and Q.

$\therefore$ KL passes through the meet of these tangents.

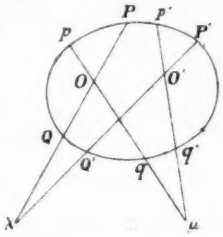
Also KL passes through the 4th harmonic to P, O, Q.

So KL passes through the meet of the tangents at A and B and the 4th harmonic to A, O, B, and these 2 points are fixed;

$\therefore$ KL is fixed.

(7) Define KL as polar of O and show by limiting cases that it is the chord of contact of tangents from O and becomes the tangent if O is on the conic.

(8) The ratio of rectangles contained by segments of intersecting chords whose directions are given is constant.



Let POQ, pOq and $P'O'Q'$, $p'O'q'$ be 2 pairs of chords, PQ meeting $P'Q'$ at λ and pq meeting $p'q'$ at μ.

Express the fact that the sides of $O\lambda O'\mu$ are cut in ratios whose product is 1, and let λ and μ move to infinity.

The proposition $PN^2 \propto AN.A'N$ follows at once for a conic which cuts its axis in 2 points A and A', and hence the double symmetry of a central conic.

Conicoids.—[The well-known geometrical proof, by involution, that one and only one conic can in general be described through 5 points will be assumed in what follows. The existence of imaginary points of intersection of curves, based on the principle of Continuity, will also be taken for granted, as it enables the proofs of some of the theorems to be greatly abbreviated.]

(9) If 2 points be taken on each of the sides of a $\triangle$, so that the product of the ratios in which these sides are cut $=1$, the 6 points of section lie on a conic.

For the conic through 5 of them must pass through the sixth, by (3).

(10) If of 3 conics C_1, C_2, C_3, lying in 3 planes which have not a common section each pair meet in 2 points, then the 6 points in which they cut any other plane lie on a conic.

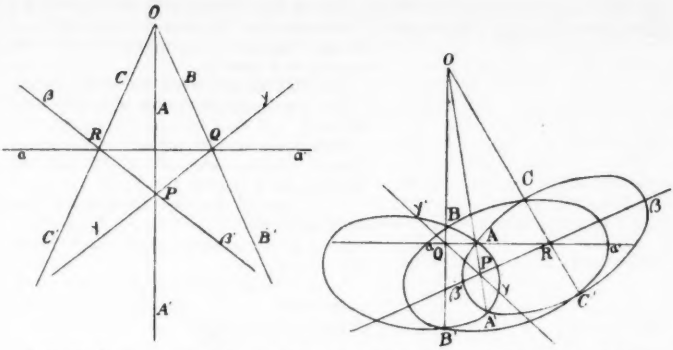
Let AA', BB', CC' be the common chords meeting at O, and let a 4th plane cut them in P, Q, R, and the conics in $\alpha, \alpha' : \beta, \beta' : \gamma, \gamma'$. Denote by $[PQ]$ the product of the ratios in which PQ is cut, and note that $[PQ][QP]=1$.

Then $[OQ][QR][RO]=1$, $[OR][RP][PO]=1$, and $[OP][PQ][QO]=1$.

Multiply $\therefore$ $[QR][RP][PQ]=1$,

i.e. $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ lie on a conic.

It is assumed that A, B, C do not coincide: they may do so as a limiting case, but the tangents to the 3 curves at A must be coplanar.



(11) If there are 4 coplanar pairs of points, $\alpha, \alpha' : \beta, \beta' : \gamma, \gamma' : \delta, \delta'$, such that every set of 3 pairs is co-conical, then either the eight points are co-conical or the joins of pairs are concurrent.

Project α, α' into the circular points at infinity. Then $\beta\beta', \gamma\gamma', \delta\delta'$ become the common chords of 3 circles and are therefore concurrent, and similarly $\alpha\alpha'$ is concurrent with them. This proof of concurrency fails if 2 of the 4 conics are coincident, and in that case the 8 points are co-conical.

(12) Let the figure of (10) stand. Denote the 4th conic by C_4. Let a 5th plane cut the planes of the 4 conics and let the conic C_5 which passes through the points where it cuts C_1, C_2, C_3 meet the plane of C_4 in δ, δ'. Then by (11), δ and δ' are on the conic C_4.

So all conics which are determined by the groups of 6 points in which planes intersect the 3 conics C_1, C_2, C_3 intersect in pairs of points and form a mutually interlacing system determining a surface. This surface is called a conicoid.

N.B.—The above points of intersection may be unreal, but they are real if the conics meet each other's planes.

(13) Every section of the surface is a conic. For by (12), all points in which it meets the plane of C_4 lie upon C_4.

Hence every line is met by the surface in not more than 2 points.

(14) The product of the ratios in which the sides of a non-coplanar polygon $A_1A_2 \ldots A_n$ are cut by a conicoid $= 1$. This is obvious if the polygon is plane and hence for a triangle.

$$\therefore\ [A_1A_2][A_2A_3][A_3A_4]\ldots[A_nA_1]$$
$$=[A_1A_3][A_3A_4]\ldots[A_nA_1]=[A_1A_4][A_4A_5]\ldots[A_nA_1], \text{ etc.}$$
$$=1.$$

Hence (8) is true for conicoids.

(15) Parallel sections are similar and similarly situated. For by (8) their parallel diameters are proportional.

(16) The locus of the harmonic conjugate of a fixed pt. O with respect to extremities of a chord through O is a plane. And this plane clearly contains the points of contact of tangent lines through O.

If O is at infinity its polar plane thus bisects all chords through O; in other words the locus of middle points of parallel chords is a plane. And if the section formed by this plane is a central conic, its centre C is a centre to the conicoid. For it is the centre of every section which contains the line CO, and hence all chords through C are bisected at C.

It is easy to deduce the elementary properties of conjugate diameters and the fact that there is always one set mutually at right angles.

Generators.—If the conicoid has a hyperbolic section, and is met by the diameter through its centre, the tangent plane parallel to that section must cut it in two straight lines since all parallel sections are similar. If this plane be made to revolve round one of these generators it will continually cut the surface in a st. line. Thus there are 2 mutually intersecting systems of generators, all those of one system being cut homographically by those of the other. Hence if a generator of one system is entirely at infinity the remainder of that system cut those of the other system proportionally and are consequently parallel to one plane. Those of the 2nd system are also parallel to one plane since they meet one and the same line at infinity.

C. A. RUMSEY.

DULWICH COLLEGE, *October*, 1903.

MATHEMATICAL NOTES.

131. [I. 25.] The following paradox may serve to illustrate M. Bertrand's remark "L'infini n'est pas un nombre" (*Calcul des Probabilités*, p. 4). It is easily exposed by adding to clause 1 the words 'not exceeding n' after the words 'integer' and 'integers':

1. Let a number mean either an integer or a fraction whose numerator and denominator are integers.
2. For every number greater than unity there is a number less than unity [its reciprocal].
3. For any number (x) less than unity there is a number ($1+x$) lying between 1 and 2.
4. Hence there are as many numbers lying between 1 and 2 as there are numbers greater than 1. Therefore there are no numbers greater than 2.

C. S. JACKSON.

132. [R. 8.] *A contrivance for shewing bending moment diagrams.*

The beautiful models of Professors Alexander and Thomson, illustrating the bending moment set up by the passage of a concentrated load or loads across a span are well known.

The following contrivance for the same purpose was suggested by these models. It can be put together in a few minutes:

A pulley Q serves as the wheel which runs along a rail AB represented by a wooden lath. A vertical rod QP hangs from the pulley. On this rod slides quite easily a collar P [a piece of rather stout copper wire with a few beads on it to act as friction rollers].

Threads PA . PB pass through small rings at AB and have small weights at their other ends.

A thread fastened at S, the middle point of AB, passes through a loop on P and is fastened to the axle of the pulley Q. As the wheel runs along the rail, P evidently traces a parabola focus S.

[For, producing QP to M making QM equal to the length of string, M lies on a fixed line parallel to AB, and

$$SP+PQ=QM=PM+PQ,$$

or

$$SP=PM.]$$

Thus it is easily seen that for any position of the wheel the threads PA . PB shew the instantaneous bending moment diagram, while the parabolic locus of P is the diagram of the maximum bending moments set up at each point in turn by the passage of the load.

A pencil twisted up in the wire will draw a very fairly good parabola.

With a little more trouble the same construction can be applied to draw the four parabolic arcs which arise when two moving loads cross the span, and of course a more finished model could be constructed if desired.

C. S. JACKSON.

133. [D. 1. b.] *Mathematical Note.*

A method of determining a very rapidly converging series for the square root of any positive integer H.

Let $x=m_1$, $y=n_1$ be any positive integral solution of the equation $x^2-Hy^2=4$.

Let $m_{r+1}=m_r^2-2$.

Denote $\frac{1}{2}(m_r-\sqrt{m_r^2-4})$ by $f(m_r)$, so that

$$\begin{aligned}1+f(m_{r+1}) &= 1+\tfrac{1}{2}(m_{r+1}-\sqrt{m_{r+1}^2-4})\\ &= 1+\tfrac{1}{2}(m_r^2-2-\sqrt{m_r^4-4m_r^2})\\ &= \tfrac{1}{2}m_r(m_r-\sqrt{m_r^2-4});\end{aligned}$$

$$\therefore\ f(m_r)=1/m_r\{1+f(m_{r+1})\};$$

$$\begin{aligned}\therefore\ f(m_1) &= 1/m_1\{1+f(m_2)\}\\ &= 1/m_1+1/m_1m_2\{1+f(m_3)\}\\ &= 1/m_1+1/m_1m_2+1/m_1m_2m_3+\ldots\end{aligned}$$

since $\operatorname{Lt}_{r=\infty} f(m_r)=0$;

$$i.e.\ \tfrac{1}{2}(m_1-n_1\sqrt{H})=1/m_1+1/m_1m_2+1/m_1m_2m_3+\ldots.$$

For instance, by putting $H=2$, $m_1=6$, $n_1=4$, we have

$$3-2\sqrt{2}=1/6+1/6.34+1/6.34.1154+1/6.34.1154.1331714+\ldots.$$

Eight terms of this series give $\sqrt{2}$ to 390 places of decimals.

G. N. WATSON.

134. [L¹. 3. a.] *Geometrical Note on the Parabola.*

"The chords AB, CD of a parabola intersect at O, and lines through C, D parallel to AB cut the diameter through O in C', D' respectively. Shew that $AO.BO=CC'.DD'$. Hence find the direction of the axis of a parabola which passes through four given points."

Peterhouse and Sidney Sussex, Dec. 17th, 1902.

Since $AO.BO:CO.DO$ = ratio of the squares on respectively parallel sides of any triangle whose base is a diameter $=CC'^2:CO^2$. Therefore

$$AO.BO:CC'^2=CO.DO:CO^2=DO:CO=DD':CC';$$

and

$$AO.BO=CC'.DD'.$$

The construction required is therefore as follows: through C draw a parallel to AB and upon it take a point C'' such that $CC'':AO.BO=CO:DO$. Then $C''O$ is the direction of the axis. [There are two directions, as C'' may be taken either side of C.]

R. F. DAVIS.

135. [D. 2. a. β.] *On the proof of Riemann's theorem on semi-convergent series.*

The proofs in Riemann's paper and Whittaker's *Modern Analysis* only indicate one way of rearranging the terms of the series so as to make it converge to any desired values, but there are an infinite number of ways.

For let the positive terms of series form series denoted by P, and let the negative terms of series form series denoted by N.

Let a', b', c' ... be any finite positive integers.

Take $a+a'$ terms of P where a is number of terms of P that must be taken to make the sum of a terms of P just $>S$.

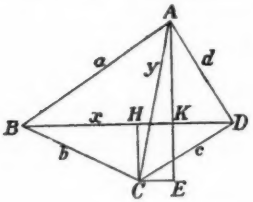
To these add $b+b'$ terms of N where b is number of terms of N that must be taken to make the sum of $a+a'$ of P, and b of N just $<S$.

To these add the next $c+c'$ terms of P where c is the number of terms of P that must be taken to make the sum of the $a+a'$ of P, $b+b'$ of N, and c of P just $>S$, and so on.

Since, if the $q'+1$ terms are from N they will be all negative, at q^{th} stage the new series has a sum differing from S by less than | the sum of $q'+1$ terms | of P (or N) where q' is the integer of series a', b', c' ... selected for q^{th} stage. But the terms of P and N are eventually infinitely small and $\therefore$ sum of a finite number of them is.

Thus the new series converges to S, and since a', b', c' were arbitrary finite integers there is a infinite number of rearrangements. H. L. TRACHTENBURG.

136. [K. 8. f.] *On the maximum quadrilateral of given sides.*



$BD=x$, $AC=y$.

(1) $x \cdot AE$ to be a maximum (twice the area);

(2) while $x \cdot CE$ = constant

$$=\frac{(a^2-d^2)-(b^2-c^2)}{2}.$$

Now, when (1) is a maximum, so is

$(1)^2+(2)^2=x^2y^2$; $\therefore$ xy is a maximum.

But $xy<ac+bd$ unless the figure is concyclic;

$\therefore$ for maximum the figure must be concyclic. R. CHARTRES.

137. [K. 4.] *Philo's Line.*

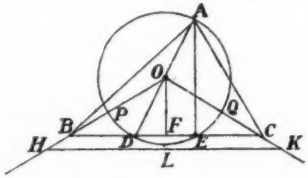
D = the fixed point.

BDC, the base of the triangle OBC, is infinite when BDC is parallel to AB or to AC;

$\therefore$ it must have a minimum value at the position of symmetry or when $OB=OC$;

$\therefore$ perp. OF bisects BC as well as DE;

$\therefore$ $BD=CE$.



Or, without using the position of symmetry, since BDC is the minimum line through D;

$\therefore$ HLK, the par^{l} tangent, is a minimum.

But the minimum tangent to arc PQ is bisected at L;

$\therefore$ BC also is bisected at F; $\therefore$ $BD=CE$. R. CHARTRES.

138. [K. 2. b, c.] *To prove that the 9-point circle touches the in- and ex-circles of a triangle.*

Let D, E, F be the points of contact of the in-circle with BC, CA, AB. A', B', C', the mid-points of the sides.

Bisect $\hat{A}$ by AU. From AC cut off $AH=AB$.

Join UH cutting $A'B'$, $A'C'$ at b', c'.

By Euc. I. 4, $\triangle AHU=\triangle ABU$ and thus $A\hat{U}H=A\hat{U}B$;

$\therefore$ UH touches the in-circle.

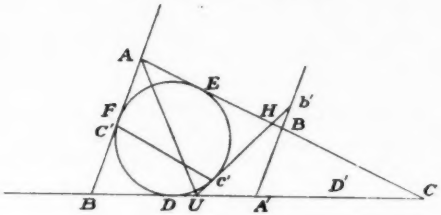
We will now prove $A'b' \cdot A'B'=A'D^2$.

$$A'D=A'B-BD=\tfrac{1}{2}a-\tfrac{1}{2}(c+a-b)=\tfrac{1}{2}(b-c).$$

Also $$\frac{BU}{c}=\frac{CU}{b}=\frac{2A'U}{b-c};\quad\therefore\quad\frac{A'U}{BU}=\frac{A'D}{c}\qquad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$A\hat{H}U=A\hat{B}U=B'\hat{A}'C;\quad\therefore\quad C\hat{H}U=b'\hat{A}'U;$$

$\therefore$ $\triangle$s $UA'b'$, CHU are similar,



and therefore $$\frac{A'b'}{CH}=\frac{A'U}{UH}=\frac{A'U}{BU}=\frac{A'D}{c}\text{ from (1),}$$

but $$CH=b-c=2A'D\text{ and }c=2A'B';$$

$$\therefore\quad A'b'\,.\,A'B'=A'D^2$$

$$=\text{(similarly) }A'c'\,.\,A'C'.\qquad\ldots\ldots\ldots(2)$$

Inverting with A' as centre and $A'D$ as radius of inversion, the straight line UH becomes by (2) the circle through $A'B'C'$, *i.e.* the 9-point circle, and the in-circle inverts into itself; but UH touches the in-circle; therefore the 9-point circle must also touch it.

UH also touches the ex-circle opposite A and $A'D'=A'D$;

$\therefore$ the 9-point circle also touches the ex-circle opposite A,

and similarly the other ex-circles. W. F. Beard.

PROBLEMS.

501. [A. 1. b.] If $abc=xyz$, and $\Sigma bc=\Sigma yz$, shew that

$$\Sigma x-\Sigma a=\frac{1}{x^2}(x-a)(x-b)(x-c),$$

and that the result may be generalized for the case of two sets of n quantities connected by $n-1$ symmetric equations. Solidus.

502. [B. 1. a.] Evaluate the determinant

$$\begin{vmatrix}1+a^2+a^5 & 1+ab+a^3b^2 & 1+ac+a^3c^2\\ 1+ab+a^2b^3 & 1+b^2+b^5 & 1+bc+b^3c^2\\ 1+ac+a^2c^3 & 1+bc+b^2c^3 & 1+c^2+c^5\end{vmatrix}\qquad\text{(O.)}$$

503. [B. 3. d.] Shew that the result of eliminating x, y, z from

$$x+y+z=a,$$
$$(y-z)^2+(z-x)^2+(x-y)^2=b,$$
$$x(y-z)^2+y(z-x)^2+z(x-y)^2=c,$$
$$x^2(y-z^2)+y^2(z-x^2)+z^2(x-y^2)=d$$

is $$b^2-2a^2b+12ac-18d=0.\qquad\text{(C.)}$$

504. [A. 1. b.] If $x+y+z=0$, then x^3y^2 can be expressed in the form

$$ax^4y+bxy^4+cxz^4+dx^4z+ey^4z+fyz^4$$

when a, b, c, d, e, f are numbers whose values are to be found. (C.)

505. [J. 2. f.] Two men whose rates of walking are 3 miles and 4 miles per hour respectively, started from a given point along the same road between 9 A.M. and 10 A.M. It is assumed that the chance that the slower walker started between x and $x+dx$ minutes past nine is $dx/60$, but nothing further is known as to their times of starting except that the faster walker did not start before the slower. Shew that the chance that the former should overtake the latter between 10.30 A.M. and 10.36 A.M. is

$$\{1+6\log_e 5-5\log_e 6\}/40. \qquad \text{(O.)}$$

506. [K. 20. e.] DEF is a triangle similar to ABC, and DE is at right angles to BC, while the vertices D, E, F lie on AB, BC, CA respectively. Prove that if a, b, c are the sides of ABC the circum-radius of DEF is

$$\frac{a^2bc^2}{2a^2c^2+b^2c^2+c^2a^2-b^4}. \qquad \text{(O.)}$$

507. [K. 11. d.] AOC, BOD are two chords of a circle: prove that the distance of the centre of this circle from the centre of either of the circles AOB, COD is equal to the radius of the other. (O.)

508. [L¹. 16. b.] Circles are described on any two perpendicular focal chords of a fixed conic as diameters: shew that their radical axis passes through a fixed point and their intersections lie on a fixed circle. (O.)

509. [R. 7. b. γ.] A particle is projected with velocity $\sqrt{2gh}$ so that its range on a horizontal plane through the point of projection is $4r$, and when it is at a distance l from the point of projection it is moving at right angles to the original direction of motion. Prove the relation $l^2r^3=h^3(l-h)$. (C.)

510. [R. 9. a.] An isosceles triangular right prism of weight W and vertical angle 2β rests with its base on a rough inclined plane and its edges horizontal: to the mid-point of the lower of the two faces, which are not in contact with the plane, is attached a light string which passes over a smooth peg and is connected to a weight w hanging freely, the non-vertical portion of the string being parallel to a line of slope of the inclined plane. Shew that for equilibrium

$$\mu>\tan\alpha+\frac{w}{W}\sec\alpha,$$

$$3w<2W\cos\alpha(3\tan\beta-\tan\alpha). \qquad \text{(C.)}$$

SOLUTIONS.

426. [L¹. 16. a.] *If the sides of a triangle are* $\sqrt{pk\pm a^2}$, $\sqrt{pk\pm b^2}$, $\sqrt{pk\pm c^2}$ *where* p *is any positive integer and* $k=\Sigma a^2$; ω_p, e_p, *the Brocard angle and eccentricity of the Brocard ellipse;* $\lambda^2=\Sigma a^2b^2$, $\nu^4=\Sigma a^4$, *prove that*

$$\cot\omega_p=(3p\pm1)k/D\,.\;\lambda_p{}^2e_p{}^2=\lambda^2e^2,$$

where $$D^2=(3p^2\pm2p)k^2+16\Delta^2;$$

also find $\cot A_p$ and $\sum_1^r(e_p{}^{-2}+e_p{}^{1-2})$, *where the accent means that the negative sign is taken in the ambiguities.* R. TUCKER.

Solution by PROPOSER.

Let the triangles $\sqrt{pk\pm a^2}$, $\sqrt{pk\pm b^2}$, $\sqrt{pk\pm c^2}$ be denoted by Δ_p, Δ'_p.

Now $k'_p=(3p\pm1)k$, $\lambda_p{}^2=(3p^2\pm2p)k^2+\lambda^2$, $\nu_p{}^4=(3p^2\pm2p)k+\nu^4$,
and $16\Delta_p{}^2=2\lambda_p{}^2-\nu_p{}^4=(3p^2\pm2p)k^2+16\Delta^2$. } .....(1)

Also $\cot\omega_p=(3p\pm1)k/\sqrt{(3p^2\pm2p)k^2+16\Delta^2}=(3p\pm1)k/D$. .....(2)

Hence $\operatorname{cosec}^2\omega_p = \{(12p^2 \pm 8p + 1)k^2 + 16\Delta^2\}/D^2$,

and $$e_p{}^2 = 1 - 4\sin^2\omega_p = (\nu^4 - \lambda^2)/[(3p^2 \pm 2p)k^2 + \lambda^2]. \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

From (1) we see that $\nu_p{}^4 - \lambda_p{}^2 = \nu^4 - \lambda^2$,

and therefore $\lambda_p{}^2 e_p{}^2 = \lambda^2 e^2$.

From (3), $e_p{}^{-2} + e'_p{}^{-2} = (6p^2k^2 + 2\lambda^2)/\lambda^2 e^2$,

and $e'_p{}^{-2} - e_p{}^{-2} = 4pk^2/\lambda^2 e^2 = 16p\cos^2\omega/e^2$;

and from (1), $4(\Delta_p{}^2 - \Delta'_p{}^2) = pk^2$.

We also get $\cot A_p = \{(p \pm 1)k \mp 2a^2\}/D$,

whence we again get (2).

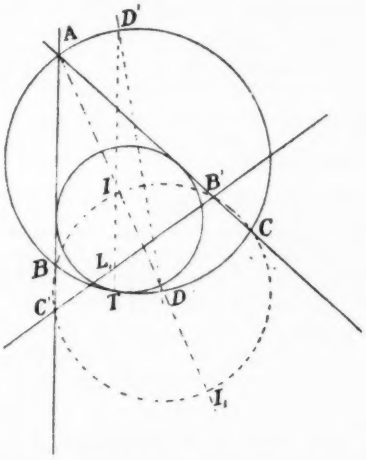
$$\sum_1^r e_p{}^{-2} = [k^2\{3\sum_1^r p^2 \pm 2\sum_1^r p\} + r\lambda^2]/(\nu^4 - \lambda^2) = \frac{r(r+1)}{2}\{(r+2\pm 2)k^2 + \lambda^2\}/(\nu^4 - \lambda^2),$$

and hence $$\sum_1^r (e_p{}^{-2} + e'_p{}^{-2}) = r(r+1)\{(r+2)k^2 + \lambda^2\}/\lambda^2 e^2.$$

427. [K. 12. b.] *Four circles can be drawn to touch two given lines AB, AC and (at T, T_1, T_2, T_3 respectively) a given circle ABC: prove this construction:—bisect the arcs BC at D, D'; with centres D, D' draw circles BC; of these draw the diameters II_1 and I_2I_3 which pass through A; the lines joining each centre to the ends of the other diameter determine T, T_1, T_2, T_3. Show also that TT_1 and T_2T_3 meet BC on the tangent at A to the circle ABC.* C. E. YOUNGMAN.

Solution by PROPOSER.

Invert from A : circle (D) into itself : B and C become C' and B' : $\odot ABC'$ line $B'C'$: the four circles, the in- and ex-circles of $AB'C'$, which touch $B'C$, at L, L_1, L_2, L_3 : and T, T_1, T_2, T_3 become L_1, L, L_3, L_2. Also D' becomes P where AD' meets $B'C'$. PI_1L_1 are concyclic with A (diameter PI_1, I_1 being excentre opposite A) ; $\therefore$ $D'IT$ are collinear.



Again, LL_1, L_2L_3, and $B'C'$ have a common mid-point ; $\therefore$ $\odot$'s ALL_1, AL_2L_3, $AB'C'$ have collinear centres and meet again at U, and AU is $\parallel B'C'$, and $\therefore$ touches $\odot ABC$; $\therefore$ T_1T, T_3T_2, and BC meet on AU.

The proof is perhaps neater if we use "symmetric inversion".—Casey's *Sequel*, 6th edition, p. 268.

428. [R. 9. b.] *Two equal smooth spheres are moving with uniform velocities U and V on a smooth horizontal table; at one instant the distance between their centres is a times the sum of their radii, and their directions of motion make angles θ and ϕ with the line joining their centres at that instant; prove that if the spheres collide the impulse between them will be*

$$I\{V^2+U^2-2UV\cos(\theta-\phi)-a^2(V\sin\phi-U\sin\theta)^2\}^{\frac{1}{2}},$$

where I is the impulse which would be produced by direct collision with unit relative velocity. Trin. (C.), 1892.

Solution by H. W. SHOEBRIDGE.

Let A, B be the positions of the centres at observed instant, C, D at instant of collision, Q the intersection of AC, BD, and E a point on BA produced. Then if t is the time elapsing between these positions,

$$AC=Ut,\quad BD=Vt,\quad AB=a\,.\,CD,\quad \angle CAE=\theta,\quad \angle DBA=\phi;$$
$$\therefore\ \angle AQB=\theta-\phi.$$

Complete the parallelogram $DCAF$. Join BF. Draw BG perpendicular to AF. Draw DK and CL perpendicular to AB, and CH perpendicular to DK.

$\therefore$ Since the spheres are smooth,

required impulse $= I\{$resolved part of relative velocity in direction $DC\}$.

Let $\lambda=$ relative velocity.

Then since $BD=Vt$, and $FD=AC=Ut$,

λt is a vector compounded of BD and DF, *i.e.* $\lambda t=BF$.

$\therefore$ Resolved part of λt parallel to $DC=GF$.

Now
$$GF^2=BF^2-BG^2,$$
$$BF^2=BD^2+DF^2-2BD\,.\,DF\cos BDF$$
$$=t^2\{U^2+V^2-2UV\cos(\theta-\phi)\}.$$

$\triangle$s AGB and CHD are similar, and $AB=a\,.\,CD$;

$$\therefore\ BG=a\,.\,DH$$
$$=a(DK-CL)$$
$$=a(Vt\sin\phi-Ut\sin\theta).$$

$$\therefore\ GF^2=t^2[\{U^2+V^2-2UV\cos(\theta-\phi)\}-a^2(V\sin\phi-U\sin\theta)^2].$$

$$\therefore\ \text{Required impulse}=I\frac{GF}{t}$$

$$=I\{V^2+U^2-2UV\cos(\theta-\phi)-a^2(V\sin\phi-U\sin\theta)^2\}^{\frac{1}{2}}.$$

432. [A. 3. a]. *The discriminant Δ of a polynomial $f(x)$ being expressed in the form $\phi(x)f(x)+\chi(x)f'(x)$, prove that when $\Delta=0$ the common factor of $f(x)$ and $f'(x)$ is also a factor of $\chi(x)$ and $\phi(x)+2\chi'(x)$.* R. W. H. T. HUDSON.

Solution by T. J. I'A BROMWICH.

Let $(x-a)$ be the repeated factor of $f(x)$, so that $f(x)$ is divisible by $(x-a)^2$. Write $x=a+y$, then we must have, on arranging in powers of y,

$$f(x)=\tfrac{1}{2}f_2y^2+\tfrac{1}{6}f_3y^3+\dots,$$
$$f'(x)=f_2y+\tfrac{1}{2}f_2y^2+\dots.$$

Assume
$$\phi(x)=\phi_0+\phi_1y+\tfrac{1}{2}\phi_2y^2+\dots,$$
$$\chi(x)=\chi_0+\chi_1y+\tfrac{1}{2}\chi_2y^2+\dots;$$

then
$$f(x)\phi(x)+f'(x)\chi(x)$$
$$=(\phi_0+\phi_1y+\tfrac{1}{2}\phi_2y^2+\dots)(\tfrac{1}{2}f_2y^2+\tfrac{1}{6}f_3y^3+\dots)$$
$$+(\chi_0+\chi_1y+\tfrac{1}{2}\chi_2y^2+\dots)(f_2y+\tfrac{1}{2}f_3y^2+\dots).$$

Now the last expression is to vanish for all values of x, and so for all values of y. Hence

$$f_2\chi_0=0,\ f_2(\phi_0+2\chi_1)+f_3\chi_0+0.$$

But, in general, $f_2 \neq 0$, and thus we have

$$\chi_0=0,\ \phi_0+2\chi_1=0,$$

which are equivalent to

$$\chi(a)=0,\ \phi(a)+2\chi'(a)=0,$$

the results to be proved.

If it happens that $f_2=0$, we may find, say

$$f_2=0,\ f_3=0,\ \dots,\ f_{m-1}=0,\ f_m \neq 0,$$

and

$$f(x)=\frac{1}{m\,!}f_m y^m+\frac{1}{(m+1)\,!}f_{m+1}y^{m+1}+\dots,$$

$$f'(x)=\frac{1}{(m-1)\,!}f_m y^{m-1}+\frac{1}{m\,!}f_{m+1}y^m+\dots.$$

Just as before, we find that

$$\chi_0=0,\ \phi_0+m\chi_1=0.$$

REVIEWS.

Lehrbuch der analytischen Geometrie in homogenen Koordinaten: von Dr. WILHELM KILLING. (Paderborn, F. Schöningh.)

Erster Teil :—**Die ebene Geometrie,** 1900, pp. xiii. and 220.

Zweiter Teil :—**Die Geometrie des Raumes,** 1901, pp. viii. and 361.

This book is, as its title suggests, a systematic introduction to the use of homogeneous coordinates, both in the plane and in space. The two parts are arranged on exactly parallel lines; in the nature of things, a greater number of cases arise for discussion in the second part, but the general sequence of ideas is the same in the two.

At various times during the past two years, I have used parts of the book in arranging courses of lectures; and it may be useful to indicate briefly some points where the author's methods seemed preferable to those employed in the ordinary English texts.

The reader will observe, early in the book, that Dr. Killing's attitude towards *homogeneous coordinates* is very different from that of the English books towards *trilinears*; it is true that his first set of coordinates is the trilinear set, but he soon passes to the general homogeneous system. In this system we take an arbitrarily selected point (not on the sides of the fundamental triangle) as the unit-point (1, 1, 1); and the system is then completely fixed. Looking at the system of coordinates in this way, we grasp more readily the fact that homogeneous coordinates are not primarily adapted for the discussion of *metrical* theorems; their proper use, at least in the first instance, is to obtain *projective* properties of figures.* Thus, as it seems to me, it is not advisable to make students calculate the trilinear coordinates of points which are defined by metrical relations to the fundamental triangle; at a later stage in their work, they can make use of the *absolute* to obtain any useful metrical theorems; but obviously the absolute should not be introduced until the student is already familiar with the use of homogeneous coordinates. Here Dr. Killing sets us an excellent example in postponing all metrical theorems to the last quarter of each part of his book.

* It may not be out of place to remark that many metrical results are often given as examples on trilinears (and areals) which can be obtained more easily by using suitably chosen Cartesian coordinates (X, Y) or complex coordinates $(x=X+iY,\ y=X-iY)$.

The question of points at infinity and the line at infinity (in a plane) is very carefully and clearly discussed by Dr. Killing: this is only what might have been expected from an author to whom the Lobatschewsky Prize was awarded for his researches on the foundations of Non-Euclidean geometry. The matter is usually handled in a very slipshod way by English authors*; thus, as a rule, no reason is given for supposing that the points at infinity form *a straight line* rather than a circle (or any other curve). But, as a matter of fact, this is one place where the difference between Euclidean geometry and Non-Euclidean (hyperbolic) geometry shows itself; the postulate of a *straight line* at infinity being equivalent to Euclid's parallel postulate (commonly called the "Twelfth Axiom" by English editors).

After discussing points at infinity in general, Dr Killing considers some special forms of homogeneous coordinates which arise when one angle (or two) of the fundamental triangle is at infinity. In the special case when two angles of the triangle are at right angles (the third being at infinity), we can, by proper choice of the unit-point, reduce the homogeneous coordinates to $(\cot\theta_1, \cot\theta_2, 1)$, where θ_1, θ_2 are the so-called "biangular coordinates."

In discussing conics and quadrics, the author pays some attention to the *degenerate* forms, so frequently overlooked in the English books; and here as everywhere else in the book, stress is laid on the *dual* treatment of conics and quadrics. That is, the properties of conics are obtained from their *line*-equations † as well as from their *point*-equations, and quadrics are discussed by the aid of *plane*-equations (commonly called *tangential* equations by English authors). Not enough space is usually devoted in our books to working with plane-equations (and line-equations); even in the applications of analytical geometry, the use of plane-equations often leads to considerable saving of labour; a noteworthy example may be found in the discussion of the constants of inertia of a rigid body.

The advantage of using plane-coordinates instead of point-coordinates is specially striking in Dr. Killing's treatment of *confocal quadrics*; in particular, the focal conics show themselves very naturally; and the confocal paraboloids can be treated without regarding them as the limit of central quadrics. I take the opportunity of remarking that nearly all the properties of a confocal central system (referred to the three normals at a given point as axes) are given by the plane-equation ‡

$$(\lambda-\lambda_1)u^2+(\lambda-\lambda_2)v^2+(\lambda-\lambda_3)w^2-2t(p_1u+p_2v+p_3w)-t^2=0;$$

for instance, the tangent-cone from the origin to any confocal is given by taking $t=0$; and the equation to the polar reciprocal can be written down at once.

There are various other points in Dr. Killing's book which could be introduced into English texts with advantage: but perhaps enough has been said to give an idea of his methods of treatment. I can recommend the book most heartily to any persons who are engaged in teaching analytical geometry.

T. J. I'a. BROMWICH.

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* I must explicitly exclude Miss C. A. Scott's excellent work from this and other criticisms.

† It may be worth while to call attention to *the advantage of graphing conics from line-equations as well as from point equations.* Thus, if a typical line (in Cartesians) is $ux+vy+1=0$ the conic whose line-equation is $v^2+au^2+2bu+c=0$, may be graphed by writing it in the form

$$\left(-\frac{1}{v}\right)^2=\left(-\frac{1}{u}\right)^2\Big/\left[-c\left(-\frac{1}{u}\right)^2+2b\left(-\frac{1}{u}\right)-a\right].$$

The intercepts $\left(-\frac{1}{u}, -\frac{1}{v}\right)$ on the coordinate axes can then be plotted by calculating $-\frac{1}{v}$ for a succession of values of $-\frac{1}{u}$. Thus a number of lines are drawn, which give a fairly good graph of the conic. Rather more care is necessary than when working from a point-equation; but graphs of this kind enable ordinary students to grasp more clearly the fact that a line-equation is just as fundamental as a point-equation.

‡ This equation is easily obtained from the results given by Dr. Killing (though the equation is not given by him); as usual, λ_1, λ_2, λ_3 are parameters of the confocals through the origin; p_1, p_2, p_3 are the coordinates of the centre of the confocals, referred to this origin; the equation to a plane is taken to be $ux+vy+wz+t=0$.

In a paraboloidal system the term t^2 disappears and p_1, p_2, p_3 are proportional to the direction-cosines of the axis of the system.

Elementary Geometry. By C. GODFREY, M.A., and A. W. SIDDONS, M.A. 3s. 6d. (Cambridge University Press.)

A New Geometry for Schools. By S. BARNARD, M.A., and L. M. CHILD, B.A. (Macmillan & Co.)

A School Geometry. Pt. III. (Circles, 1s.) By H. S. HALL and F. H. STEVENS. (Macmillan & Co.)

Theoretical Geometry for Beginners. Pt. II. 1s. 6d. By C. H. ALLCOCK. (Macmillan & Co.)

Elementary Geometry. Sect. II. 1s. 6d. By F. R. BARRELL. (Longmans, Green & Co.)

A First Book on Geometry. 9d. By W. AHRENS, M.A., and J. D. PAUL, B.A. (Meiklejohn and Holden.)

Preliminary Geometry. By G. H. WYATT, B.Sc. (Relfe Bros.)

The work by Messrs. Godfrey and Siddons well deserves the attention it will naturally receive from the prominent part taken by the authors in the movement for securing reform in geometrical teaching. It is an excellent treatise and shows throughout its pages the marks of the enthusiastic teacher who loves his subject. An experimental portion (55 pages) precedes the theoretical (300 pages). In arrangement the authors follow the schedule of geometry recently adopted at Cambridge, which, we are glad to notice, has been widely circulated by the Vice-Chancellor. They have given a profusion of concrete examples well calculated to lead the young student to recognise that there is geometry everywhere, and not merely between the covers of a school-book. Among these we notice the forms of constellations given as an exercise on graphs, and the motion of a point on the connecting rod of a steam engine as an exercise on loci. But while the many and varied illustrations give the book a very different aspect from the pages of the Potts and Todhunter of our youth, there is no decline from Euclidean vigour in the formal proofs, though a well-chosen notation and use of symbols enable these to be given concisely. We are reminded in this respect of the way in which a certain Car. Robotham 'dropped into poetry,' addressed "Ad amicissimum virum J. B. *de Euclide contracto*," on the appearance of Barrow's *Euclid*:

> "Immensa dudum margo commentarii
> Diagramma circuit minutum; utque insula
> Problema breve natabat in vasto mari
> Sed unda jam detumuit et glossa arctior
> Stringit theoremata."

To one diagram, however, we must take exception, that for Perigal's dissection proof of I. 47. Here the beautiful simplicity of the diagram as published by Perigal is lost by a reversal of the large square so that the lines which dissect it are no longer parallel to the sides of the other two squares. This detracts from its effectiveness.—"The New Geometry for Schools" is also a most excellent treatise, to be added promptly to the library of every reforming teacher. It is written on much the same lines as the above, and has the same good qualities as to excellent type and a profusion of well-executed diagrams and concrete illustrations of general principles. Its scope, however, is wider, extending to such higher matters as Pole and Polar, Maxima and Minima. A pretty solution on p. 207 of the problem '*about a given quadrilateral ABCD describe a quadrilateral similar to a given quadrilateral*' is typical of the treatment. As in the above, homothetic figures are adequately treated, and their properties are applied to construction.—Pt. III. of "A School Geometry" gives the essentials of Euclid III. and IV. It is excellently adapted for its purpose, which seems to be to get the full advantage of the recent changes in the regulations while departing as little as possible from Euclidean treatment. Numerical problems are given, to which the answers are furnished. Type, printing, and figures are excellent. The division of the complete work into short parts, published separately, seems useful.—To Mr. Allcock's Pt. II. the same remarks apply. The collection of exercises is extensive and seems well selected.—Mr. Barrell's Section II. deals with the subject matter of Euclid's II. and those parts of III. which deal with the equality of areas; with

parts of IV. and VI.; with Trigonometric Ratios and the measurement of circles. It has the same merits as the two parts of Section I., reviewed in the May number of the *Gazette*. The diagrams have been designed and executed with great care, shading and algebraic symbols being used to bring out their properties effectively. The property of the centre of similitude is well illustrated by a diagram on p. 200 showing two maps of England, with London for their centre of similitude.—The small price of the Primer by Messrs. Ahrens and Paul, and the excellence of its contents, seem likely to cause it to find favour with those teachers who prefer that even the lowest classes should have a text-book to guide their efforts. While some teachers prefer at this stage to depend on oral instruction there are others of a contrary opinion. To such it can be heartily recommended, and those who make their own course may find it a useful guide in planning one.—For the latter purpose Mr. Wyatt's book might also be profitably employed. The following (p. 68) is decidedly unsatisfactory and likely to do harm:—"*Definition.* If a straight line falling on two other straight lines make the exterior angle equal to the interior and opposite angle on the same side, these two straight lines shall be parallel to one another."

First Stage Practical Plane and Solid Geometry. By G. F. Burn. (W. B. Clive & Co.)

This little book forms a useful addition to the Organized Science Series. The first seventy-two pages deal with ordinary problems of plane geometry, in which not much originality of treatment is to be expected or indeed desired. The next eleven pages treat of vectors. The remainder is devoted to Solid Geometry, and forms the strong feature of the book. There are excellent photographic reproductions of arrangements of strings and sheets of cardboard, illustrating problems of Solid Geometry and suggesting various useful contrivances to those whose funds do not allow of obtaining elaborate scientific plant, not always the most effective, by the way, for educational purposes. A set of Examination Papers at the end, with Answers, will be found extremely useful by students and teachers.

Factors in Algebra. 2s. By W. Mudie. (Oliver & Boyd.)

Besides a large collection of examples on factorising expressions of the 2nd, 3rd, and 4th degrees, this little book deals with Geometrical Progression, Surds, and Quadratic Equations. The sets of Examples are valuable. We notice that in factorising ax^2+bx+c, when both a and c differ from unity, no account is given of the method by means of choosing $pq=ac$, $p+q=b$, which seems the easiest. See D. E. Smith's *The Teaching of Elementary Mathematics*, p. 194.

Elementary Algebra. Pt. I. By Chintamani Muckerjee, B.A. (The Indian Press, Allahabad.) A nice little work of about 200 pages, not going beyond equations of the first degree, but dealing with symmetry, cyclic order, and the remainder theorem. It has excellent sets of examples. E. M. Langley.

Quaternions and Projective Geometry. By Prof. C. J. Joly. *Royal Soc. Phil. Trans.*, Vol. 201, pp. 223-327.

The perusal of these hundred quarto pages suggests a consideration of the present state of quaternion theory and the peculiar aspects with which it is regarded by different classes of mathematicians. It is difficult to say which is more deplorable, the total ignorance of some, or the disproportionate enthusiasm of others. It is surely possible to admire the elegance and brevity with which, in particular, certain kinematical theorems can be expressed, to appreciate the isotropy on which Hamilton laid so much stress, and at the same time to recognise that the special nature of quaternions and the clumsiness of much of the notation connected with them impose stringent limitations on their useful application. Quaternions should be regarded initially from the algebraic standpoint; the geometrical properties easily follow as the interpretation of the results of permissible algebraic processes. In this view, quaternions occupy a small niche in the general theory of complex numbers; they are particularised by the integer four and by the special relations among the complex units, and hence must be necessarily of limited application and lead to results capable of but slight generalisation. From the utilitarian point of view quaternions have little to justify their extensive use. Except in a

few special problems, nothing is gained by their use that is not as conveniently and effectively attained by the more natural and more general notation of matrices. Both notations aim at dealing with many quantities at once, but in the quaternion notation these quantities are bound together with chains which have to be broken by the introduction of clumsy adventitious symbols, whereas in the matrix notation there is nothing superfluous. For instance, $\Sigma p_r q_r$ is in matrix notation simply pq, but when p and q are quaternions we have to multiply one by the conjugate of the other, and then pick out the scalar part of the product, writing it Spq' or $Sp'q$.

The memoir under review falls roughly into two parts; the first half gives the theory of the four-rowed square matrix in the language of quaternions. We are introduced to the *linear quaternion function*, which has the same effect as the general linear substitution in four variables. If A is the matrix of the coefficients in this substitution, and A' is its conjugate, then obviously $A+A'$ is symmetric, and $A-A'$ is skew. Thus we are led to associate a quadric surface and a linear complex, which, with the invariant tetrahedron of the substitution, form the materials for a considerable amount of elementary geometry.

It is well known that the classification of linear substitutions A depends on the invariant factors of $|A-tE|$ where E is the matrix of the identical substitution, and that the type of intersection of two quadrics

$$\Sigma x_r A\ x_s = 0, \quad \Sigma x_r B_{rs} x_s = 0,$$

depends upon the nature of the invariant factors of $|A-tB|$. All this is carried out in quaternion notation, with the appropriate geometrical significance. The latter part of the memoir is devoted to the *bilinear quaternion function*, which is a linear combination, with quaternion coefficients, of four ordinary bilinear forms in two sets of variables. These complicated expressions are manipulated with wonderful skill, and made to yield a vast number of important geometrical results. A particularly interesting application is to the general quadratic transformation of space, in which any plane is transformed into a Steiner's quartic surface.

R. W. H. T. H.

CORRESPONDENCE.

18 Belgrave Square, Monkstown,
Co. Dublin, *November 9th*, 1903.

To the Editor of the "Mathematical Gazette."

Dear Sir,—May I call the attention of your readers to the fact that a Committee of the Education Section of the British Association was appointed, with Sir Philip Magnus as Chairman, this year at Southport, "to report upon the courses of experimental, observational, and practical studies most suitable for elementary schools," and to solicit their assistance in the important work the Committee has before it.

It is very desirable that the Committee should be in possession of all available information as to schemes of work that come within the scope of its enquiry, and which are at the present time in operation. If teachers who are carrying out original schemes of instruction, or who are acquainted with particularly good efforts in teaching observational and practical subjects, will favour the Committee with full details of such courses of instruction, it will be of the greatest possible assistance.

The enquiry will cover the following sections, and will be particularly concerned with the coordination of these with one another and with the ordinary subjects of the curriculum of an elementary school:

1. Practical and Experimental Arithmetic and Geometry.
2. Elementary Experimental Science (fundamental principles of Chemistry and Physics).
3. Nature Study and its relation to Botany and Geography.
4. Domestic Science and Art, including Cookery, Laundry-work, Housewifery, Hygiene and Needlework.

5. Manual instruction other than in the forms comprised by preceding sections, *e.g.*, Kindergarten, "Hand and Eye" training, Drawing, and the use of tools.

Teachers willing to assist the Committee will best do so by sending as full details as possible of courses of instruction, conditions of work, time devoted to the subject, methods and organisation, and (if convenient), *average* samples of the work accomplished or note books produced by the pupils; such samples will be carefully preserved, and when examined will be returned to the source from which they came.

As there is little time in which to make such an extensive enquiry, I wish, on behalf of the Committee, to appeal to teachers, inspectors, and others interested in these matters, to send to me at the above address, before Christmas, contributions that may be of service to the Committee.

Thanking you, Sir, in anticipation for the privilege of using your columns, believe me, yours very truly,

W. MAYHOWE HELLER, *Secretary to the Committee.*

NOTICES.

Professor Hudson's Saturday morning Lectures, to Teachers on the teaching of Mathematics, are postponed till next term, beginning January 23, 1904, 10 a.m., at King's College.

The Librarian begs to acknowledge receipt of the Annual Report of the "South Australian School of Mines."

COLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

(1) **For Sale.**

The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 1874 to Nov. 1882. Vols. I-IX. Edited and Published by E. HENDRICKS, M.A., Des Moines, Iowa, U.S.A.

[With Vols. V.-IX. are bound the numbers of Vol. I. of *The Mathematical Visitor.* 1879-1881. Edited by ARTEMAS MARTIN, M.A. (Erie. Pa.)].

The Mathematical Monthly. Vols. I.-III. 1859-1861 (interrupted by the Civil War, and not resumed). Edited by J. D. RUNKLE, A.M.

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Tortolini's Annali. Vol. I. (1850), or any one of the first eight parts of the Volume.

(3) Dr. Muir, The Education Office, Cape Town, will give Vol. 109, *Crelle's Journal,* to any member of the Mathematical Association whose set is without it.

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VIRGO